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On Chebyshev–Markov Rational Functions over Several Intervals

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Chebyshev–Markov rational functions are the solutions of the following extremal problem

$$\min_{c_1, ..., c_n \in \mathbb{R}} \left\| \frac{x^n + c_1 x^{n-1} + \cdots + c_n}{\omega_n(x)} \right\|_{C(K)}$$

with K being a compact subset of $\mathbb{R}$ and $\omega_n(x)$ being a fixed real polynomial of degree less than n, positive on K.

A parametric representation of Chebyshev–Markov rational functions is found for $K = [b_1, b_2] \cup \cdots \cup [b_{2p-1}, b_{2p}]$, $-\infty < b_1 \leqslant b_2 < \cdots < b_{2p-1} \leqslant b_{2p} < +\infty$ in terms of Schottky–Burnside automorphic functions. © 1998 Academic Press

1. INTRODUCTION

Let K be a compact subset of the real line, C its complement $\mathbb{R} \backslash K$, and let $\Phi = \{\phi_0, \phi_1, ..., \phi_n\}$ be a complete Tchebysheff (CT-) [15] system of continuous functions over K. The unique polynomial which deviates least from zero on K with respect to the *sup*-norm among all polynomials of the form $c_0\phi_0(x) + \cdots c_{n-1}\phi_{n-1}(x) + \phi_n(x)$, $c_i \in \mathbb{R}$, is called Chebyshev polynomial and denoted by $T_n(K, \Phi, x)$, i.e.,

$$\| T_n(K, \Phi, x) \|_{C(K)} = \min_{c_i \in \mathbb{R}} \| c_0\phi_0 + \cdots + c_{n-1}\phi_{n-1} + \phi_n \|_{C(K)}. \tag{1}$$

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For the cases $K=[-1,1]$,

$$\Phi=\Phi_P=\{1, x, ..., x^n\} \qquad \text{and} \qquad \Phi=\Phi_R=\left\{\frac{1}{\omega_n(x)}, \frac{x}{\omega_n(x)}, ..., \frac{x^n}{\omega_n(x)}\right\},$$

where $\omega_n(x)\in\mathbb{P}_n$ is a fixed real polynomial of degree less or equal n nonvanishing on K, those polynomials were found by P. L. Chebyshev [36]. In 1906 A. A. Markov (see [21]) gave another representation of $T_n(K, \Phi, x)$ by trigonometric functions for the same cases and also for $K=[-1,1]$,

$$\Phi=\Phi_A=\left\{\frac{1}{(\omega_{2n}(x))^{1/2}}, \frac{x}{(\omega_{2n}(x))^{1/2}}, ..., \frac{x^n}{(\omega_{2n}(x))^{1/2}}\right\},$$

where $\omega_{2n}(x)\in\mathbb{P}_{2n}$ is a fixed polynomial which is positive on K.

The case $K=[-1,1]$ and $\Phi=\Phi_P$ is well-known under the name "classical Chebyshev polynomials." There are many works about them and their applications, including at least four specialized books [14, 23, 30, 33].

The case $K=[-1,1]$, and $\Phi=\Phi_R$ is known in Russian mathematical literature as "Chebyshev–Markov rational functions." They have many applications in analysis and techniques (see [31] which is devoted to them, and [8, 11, 19]).

S. N. Bernstein [7] showed that those functions are orthogonal with corresponding weight on $[-1,1]$ and used them for investigations on asymptotic behavior of orthogonal polynomials relative to general weights.

The case of disconnected sets is more complicated. For system $\Phi=\Phi_P$ and $K=[-1,a]\cup[b,1]$, $-1<a<b<1$ the problem was solved by N. I. Achieser [1–4]. Interest to the problem rose after works [24, 34], where connection was discovered of Achieser's polynomials with orthogonal polynomials. Let us indicate here recent note [22], where geometric aspects of the problem are investigated. Analogue of Achieser's solution for $\Phi=\Phi_R$ and $K=[-1,a]\cup[b,1]$ was found by the author [20]. Many aspects of this problem and connection with orthogonal rational functions are contained in [27].

The case of several intervals and $\Phi=\Phi_P$ was treated recently in many works (see, for instance, [5, 38], surveys [26, 35]). One of the main advantages here is the connection with orthogonal polynomials, discovered by F. Peherstorfer, M. L. Sodin and P. M. Yuditsiĭ [25, 35]. For $\Phi=\Phi_R$ and $K=[b_1,b_2]\cup\cdots\cup[b_{2p-1},b_{2p}]$ the connection with orthogonal rational functions was discovered by F. Peherstorfer [25]. The works [18, 26, 29] contain many related results.

The main goal of the paper is to present a complete solution of (1) for any system of poles $\{1/a_{i,n}\}_{i=1}^{l_n}\subset\mathbb{C}\backslash K$ with $\omega_n(x)=\prod_{i=1}^{l_n}(1-a_{i,n}x)^{m_{i,n}}\in\mathbb{P}_{n-1}$ in terms of automorphic Schottky–Burnside functions. The method is a generalization of N. I. Achieser's method [1, Kapitel V].

At first let us remember the Chebyshev–Markov rational functions:

$$M_n(x) = M_n \cos \sum_{i=1}^{l_n} \left(m_{i,n} \arc \cos \frac{x - a_{i,n}}{1 - a_{i,n} x} + \kappa_n \arc \cos x \right),$$
$$M_n = 1 \Big/ \left(2^{\kappa_n/2 - 1} \prod_{i=1}^{l_n} (1 + (1 - a_{i,n}^2)^{1/2})^{m_{i,n}} \right). \tag{2}$$

Here and everywhere later $\Phi = \Phi_R$ with $\omega_n(x) = \prod_{i=1}^{l_n} (1 - a_{i,n})^{m_{i,n}}$, $\sum_{i=1}^{l_n} m_{i,n} + \kappa_n = n$, $\omega_n(x) \in \mathbb{P}_n$, moreover $\omega_n(x) > 0$, $x \in [-1, 1]$, and $\mathfrak{A} = \{a_{i,n}\}_{i=1, n=1}^{l_n, \infty}$ is the matrix of the inverse values of the poles, $a_{i,n} \neq a_{j,n}$, $i, j = 1, ..., l_n$, $i \neq j$.

Further we shall use the following definitions from the theory of automorphic Schottky–Burnside functions. Denote by $G(K_1, ..., K_{q-1}) \subset \mathbb{C}$ any domain which is the upper half of the complex plane without disjoint circles $K_1, ..., K_{q-1}$, lying inside it. The domain $G(K_1, ..., K_{q-1})$ together with a domain symmetric to it with respect to the real axis and with the real axis is called the fundamental domain of a Schottky group Γ (together with $\partial K_1, ..., \partial K_{q-1}$, see [13]). Generators of the group Γ are maps $T_i(z) = (R_i^2/(z - o_i)) + \bar{o}_i$, $i = 1, ..., q-1$, where o_i is a center and R_i is a radius of the circle K_i, $i = 1, ..., q-1$. The group Γ consists of mappings

$$\Gamma = \{T_i\}_{i=0}^{\infty}, \qquad T_0(z) \equiv z.$$

Now we introduce the following W. Burnside's functions [9, 10] (cf. [6, Ch. 14]):

$$\Omega(z, y) = (z - y) \prod_i{}' \frac{(T_i(z) - y)(T_i(y) - z)}{(T_i(z) - z)(T_i(y) - y)}, \tag{3}$$

$$\exp \Phi_i(z) = \frac{z - c_i}{z - c_{i^{-1}}} \prod_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{z - c_{j^{-1}i}}{z - c_{j^{-1}}}. \tag{4}$$

Here and everywhere later $c_j = T_j^{-1}(\infty)$, $c_{i^{-1}} = T_i(\infty)$, and $c_{i^{-1}j}$ equals $T_i(T_j^{-1}(\infty))$, and prime near signs of products means that of each pair of inverse substitutions T and T^{-1}, only one is to be taken in the infinite product and $i > 0$. Moreover, let

$$[z; \xi] = \prod_{i=0}^{\infty} \frac{z - T_i(\xi)}{z - T_i(-\xi)} \tag{5}$$

be J. Kluyver's function [16].

DEFINITION 1. We shall say that the nth row of the matrix $\mathfrak{A}$ is regular with respect to $K=[b_1, b_2] \cup \cdots \cup [b_{2q-1}, b_{2q}]$ if the solution R_n^* of problem (1) with $\Phi=\Phi_R$ is such that for any α with $-\|R_n^*\|_{C(K)} \leqslant \alpha \leqslant \|R_n^*\|_{C(K)}$ all zeroes of the function $R_n^*(x)-\alpha$ belong to K. The matrix $\mathfrak{A}$ is called regular with respect to K if its rows are regular for any $n \in \mathbb{N}$, $n \geqslant q$.

2. AUXILIARY RESULTS

It is more convenient from the beginning to treat the problem as one of the uniform approximation with the weight $s \in C(K)$, $s(x) \neq 0$ for $x \in K$, $K=[b_1, b_2] \cup \cdots \cup [b_{2q-1}, b_{2q}]$ of the function $f \in C(K)$ by algebraic polynomials of degree no more than n.

PROPOSITION 1 (The Chebyshev Alternation Theorem). *Let $f \in C(K)$, $E_n=\max_{x \in K} |(f(x)-p(x))/s(x)|$, and $\varepsilon(x)=(f(x)-p(x))/s(x)$. Then $p(x)$ is the best approximation of f with respect to $\mathbb{P}_n$ iff there are at least $n+2$ points $\{x_i\}_{i=1}^{n+2} \subset K$, $x_i<x_{i+1}$ such that the following relations hold*:

$$|\varepsilon(x_i)|=E_n, \qquad i=1, 2, ..., n+2, \tag{6}$$

$$\varepsilon(x_{i+1})=\varepsilon(x_i)(-1)^{1+\sum_{j \in J} s_j}, \tag{7}$$

with $J=\{j\colon x_i \leqslant b_{2j}<b_{2j+1} \leqslant x_{i+1}\}$, $(-1)^{s_j}=sign(s(b_{2j})\, s(b_{2j+1}))$.

The proof of this proposition is quite analogous to usual (see, for instance, [12]) and is omitted here.

LEMMA 1. *A row $\mathfrak{A}_n$ is regular with respect to $K=[b_1, b_2] \cup \cdots \cup [b_{2q-1}, b_{2q}]$ iff the rank of the following matrix equals $q-1$ with some $n_i \in \mathbb{N}$, $i=1, ..., q$; $n_1+\cdots+n_q=n$:*

$$\operatorname{rank} \begin{pmatrix} e_{ij} & d_i \\ f_{kj} & g_k \end{pmatrix}_{i=1, k=1, j=1}^{q, q-1, q-1}=q-1, \tag{8}$$

where

$$e_{i, j}=\int_{b_{2i-1}}^{b_{2i}} \frac{x^{q-j-1}}{(-h(x))^{1/2}}\, dx, \qquad i=1, ..., q; \quad j=1, ..., q-1; \tag{9}$$

$$f_{k, j}=\int_{b_{2k}}^{b_{2k+1}} \frac{x^{q-j-1}}{h^{1/2}(x)}\, dx, \qquad k=1, ..., q-1; \quad j=1, ..., q-1; \tag{10}$$

$$d_i = n_i \pi (-1)^{q-i+1-\lambda_i} + \kappa_n \int_{b_{2i-1}}^{b_{2i}} \frac{x^{q-1}}{(-h(x))^{1/2}}\, dx$$

$$+ \sum_{j=1}^{l_n} m_{j,n} a_{j,n} h^{1/2}(1/a_{j,n}) \int_{b_{2i-1}}^{b_{2i}}$$

$$\times \frac{dx}{(1-a_{j,n}x)(-h(x))^{1/2}}, \qquad i = 1, ..., q; \tag{11}$$

$$g_k = \kappa_n \int_{b_{2k}}^{b_{2k+1}} \frac{x^{q-1}}{h^{1/2}(x)} + \sum_{j=1}^{l_n} m_{j,n} a_{j,n} h^{1/2}(1/a_{j,n})$$

$$\times \int_{b_{2k}}^{b_{2k+1}} \frac{dx}{(1-a_{j,n}x)\, h^{1/2}(x)}, \qquad k = 1, ..., q-1; \tag{12}$$

$$h(x) = \prod_{j=1}^{2q} (x - b_j); \tag{13}$$

$$\lambda_i = \sum_{j=i}^{q} \sum_{k:\, (1/a_{k,n}) \in (b_{2j}, b_{2j+1})} m_{k,n}, \qquad b_{2q+1} = +\infty,$$

and in the nominators branches of square roots are chosen in such a way that

$$\lim_{x \to \infty} \frac{h^{1/2}(x)}{x^q} = 1,$$

in the denominators the arithmetical roots are taken, and for $1/a_{j,n} \in [b_{2k}, b_{2k+1}]$, $k = 1, ..., q$, *the corresponding integral in* (13) *is understood in Cauchy's principal value sense.*

Proof. Let $\mathfrak{A}_n$ be regular with respect to K. On K the function $R_n^*(x)$ does not exceed the value $R_n = \| R_n^* \|_{C(K)}$ and simultaneously the inequality $|R_n^*(x)| > R_n$ holds on $\mathbb{R} \setminus E$.

Assume firstly that none of the $1/a_{j,n}$'s, $j = 1, ..., l_n$ belongs to the interval $[b_1, b_{2q}]$. Then there exist numbers $n_1, ..., n_q \in \mathbb{N}$ such that $n_1 + \cdots + n_q = n$ and exactly n_j zeroes of the function $R_n^*(x)$ and $n_j + 1$ deviation points (where $|R_n^*(x)| = R_n$) belong to $[b_{2j-1}, b_{2j}]$. Hence the following relations for the function

$$f(z) = R_n^{*\prime}(z) / (R_n^{*2}(z) - R_n^2)^{1/2}$$

hold:

(1)

$$\int_{\delta_j} f(z)\, dz = \ln(R_n^*(z) + (R_n^{*2}(z) - R_n^2)^{1/2})|_{\delta_j} = 2 n_j \pi i, \tag{14}$$

where δ_j is a contour surrounding $[b_{2j-1}, b_{2j}]$ and such that no poles $1/a_{k,n}$, $k=1, ..., l_n$, lie inside (on the upper sheet of $\mathscr{R}$) and on the contour δ_j.

(2) $\int_{\gamma_i} f(z)\,dz=0$, $i=1, ..., q-1$, where γ_i is a contour coming from the upper side of the barrier $[b_{2i-1}, b_{2i}]$ on the upper sheet, then passing through the barrier $[b_{2i+1}, b_{2i+2}]$ onto the lower sheet, and coming back to the barrier $[b_{2i-1}, b_{2i}]$ such that no poles $1/a_{k,n}$, $k=1, ..., l_n$ lies "inside" and on the contour γ_i, $i=1, ..., q-1$. Moreover, $\gamma_i \cap \gamma_j = \varnothing$ for $i \neq j$, $\delta_i \cap \delta_j = \varnothing$ for $i \neq j$.

(3)

$$f(z)=\left(\kappa_n u_{q-1}(z)+\sum_{j=1}^{l_n} \frac{m_{j,n} a_{j,n} h^{1/2}(1/a_{j,n})}{1-a_{j,n} z}\right) \Big/ h^{1/2}(z), \tag{15}$$

where $u_{q-1}(z)$ is a monic polynomial of degree $q-1$.

Now let the polynomial $u_{q-1}(z)$ be a unique polynomial determined by the equations

$$\int_{\gamma_i} \left(\kappa_n u_{q-1}(z)+\sum_{j=1}^{l_n} \frac{m_{j,n} a_{j,n} h^{1/2}(1/a_{j,n})}{1-a_{j,n} z}\right) \Big/ h^{1/2}(z)\,dz=0, \qquad i=1, ..., q-1. \tag{16}$$

Hence substitution of (15) into (14) gives the relations

$$\int_{\delta_k} \left(\kappa_n u_{q-1}(z)+\sum_{j=1}^{l_n} \frac{m_{j,n} a_{j,n} h^{1/2}(1/a_{j,n})}{1-a_{j,n} z}\right) \Big/ h^{1/2}(z)\,dz = 2n_k \pi i, \qquad k=1, ..., q. \tag{17}$$

One obtains after contraction of contours γ_i and δ_k to intervals of the real axis the following equalities:

$$\int_{b_{2i}}^{b_{2i+1}} \left(\kappa_n u_{q-1}(x)+\sum_{j=1}^{l_n} \frac{m_{j,n} a_{j,n} h^{1/2}(1/a_{j,n})}{1-a_{j,n} x}\right) \Big/ h^{1/2}(x)\,dx = 0, \qquad i=1, ..., q-1, \tag{18}$$

$$\int_{b_{2k+1}}^{b_{2k+2}} \frac{\kappa_n u_{q-1}(x)+\sum_{j=1}^{l_n} \frac{m_{j,n} a_{j,n} h^{1/2}(1/a_{j,n})}{1-a_{j,n} x}}{(h(x))^{1/2}}\,dx = n_{k+1} \pi (-1)^{q+1-k-\lambda_k}, \qquad k=0, ..., q-1. \tag{19}$$

Conversely from (18), (19) it follows (16), (17) for any contours δ_i, γ_i sufficiently close to the real axis, so the function

$$\phi(z)=\exp\left(\int_{b_1}^{z}\frac{\kappa_n u_{q-1}(z)+\sum_{j=1}^{l_n}\frac{m_{j,n}a_{j,n}h^{1/2}(1/a_{j,n})}{1-a_{j,n}z}}{h^{1/2}(z)}\,dz\right)$$

is a single-valued analytic function on the Riemann surface $\mathscr{R}$ with poles in ∞_2 of order κ_n and in $(1/a_{j,n})_2$ of order $m_{j,n}$, $j=1, ..., l_n$. Indeed, it is enough to apply the Argument Principle with a sufficiently great contour C on the upper sheet surrounding the set K and all poles $(1/a_{j,n})_2$, $j=1, ..., l_n$. We find then

$$\frac{1}{2\pi i}\int_C\frac{\phi'(z)}{\phi(z)}\,dz=\frac{1}{2\pi i}\int_C\left(\kappa_n u_{q-1}(z)+\sum_{j=1}^{l_n}\frac{m_{j,n}a_{j,n}h^{1/2}(1/a_{j,n})}{1-a_{j,n}z}\right)\Big/ h^{1/2}(z)\,dz$$

$$=\operatorname*{res}_{z=\infty_2}\frac{\kappa_n u_{q-1}(z)}{h^{1/2}(z)}=-\kappa_n. \tag{20}$$

The same reasoning shows that

$$\frac{1}{2\pi i}\int_{C_j}\frac{\phi'(z)}{\phi(z)}\,dz=\operatorname*{res}_{z=(1/a_{j,n})_2}\frac{m_{j,n}a_{j,n}h^{1/2}(1/a_{j,n})}{(1-a_{j,n}z)\,h^{1/2}(z)}$$

$$=-m_{j,n},\qquad j=1, ..., l_n, \tag{21}$$

where C_j, $j=1, ..., l_n$ are disjoint small circles with centers $1/a_{j,n}$, $j=1, ..., l_n$ without intersections with K.

The form of the function $\phi(z)$ and (20), (21) show immediately that $\phi(z)$ has zeros at ∞_1 of order κ_n and at $(1/a_{j,n})_1$ of order $m_{j,n}$, $j=1, ..., l_n$. Hence the function $\phi(z)$ may be presented in the form

$$\phi(z)=R_1(z)+R_2(z)\,h^{1/2}(z),$$

where R_1 and R_2 are rational functions with poles at ∞ of order κ_n and at $1/a_{j,n}$ of order $m_{j,n}$, $j=1, ..., l_n$. Since $\phi(z)$ changes according rule $\phi(\imath(z))=1/\phi(z)$ under the involution $\imath(z)$ which interchanges sheets of the surface $\mathscr{R}$, we find

$$(R_1(z)+R_2(z)\,h^{1/2}(z))(R_1(z)-R_2(z)\,h^{1/2}(z))=1$$

or

$$(P_1(z)+P_2(z)\,h^{1/2}(z))(P_1(z)-P_2(z)\,h^{1/2}(z))=\prod_{j=1}^{l_n}(1-a_{j,n}z)^{2m_{j,n}},$$

where P_1 and P_2 are polynomials of degree $2n$ and $2n-q$ correspondingly.

Let us prove that

$$\frac{P_1(z)}{\prod_{j=1}^{l_n}(1-a_{j,n}z)^{m_{j,n}}}=C_1R_n^*(z). \tag{22}$$

First of all, $P_1(z)$ is a real polynomial. Indeed,

$$\begin{aligned} R(z)&=\frac{P_1(z)}{\omega_n(z)}=\frac{\phi(z)+1/\phi(z)}{2}\\ &=\cosh\left(\int_{b_1}^{z}\left(\kappa_n u_{q-1}(z)+\sum_{j=1}^{l_n}\frac{a_{j,n}h^{1/2}(1/a_{j,n})}{1-a_{j,n}z}\right)\Big/h^{1/2}(z)\right)dz\end{aligned}$$

and it is enough to observe that for instance $R(z)$ is a real rational function with $|R(z)|\geqslant 1$ when $z\in(-\infty,b_1)_2$. Let us check now the alternation property of $R(x)$ on K. Since

$$\begin{aligned} R(x)&=\cosh\left(\left(\left(\int_{b_1}^{b_2}+\cdots+\int_{b_{2j-2}}^{b_{2j-1}}+\int_{b_{2j-1}}^{x}\right)\right(\kappa_n u_{q-1}(x)\right.\\ &\qquad\left.+\sum_{j=1}^{l_n}\frac{m_{j,n}a_{j,n}h^{1/2}(1/a_{j,n})}{1-a_{j,n}x}\Big)\Big/h^{1/2}(x)\,dx\right)\\ &=\cosh\left(i\sum_{k=1}^{j-1}2n_k\pi+i(-1)^{q-j}\int_{b_{2j-1}}^{x}f(x)\,dx\right)\\ &=\cos\int_{b_{2j-1}}^{x}f(x)\,dx\end{aligned}$$

for $x\in[b_{2j-1},b_{2j}]$, it follows that $|R(x)|\leqslant 1$ for $x\in K$. Moreover, the equality

$$\int_{b_{2j-1}}^{b_{2j}}f(x)\,dx=n_j\pi(-1)^{q-j-\lambda_j},$$

implies the existence of n_j+1 points $e_{j,0}=b_{2j-1}<\cdots<e_{j,n_j}=b_{2j}$ with $|R(e_{j,k})|=1$ and $R(e_{j,k+1})=-R(e_{j,k})$, $j=1,...,q$; $k=0,1,...n_j$. So the points $e_{1,0},...,e_{1,n_1},e_{2,0},...e_{q,n_q}$ form an alternant, and the case if all poles $1/a_{j,n}$ are outside of $[b_1,b_{2q}]$ is proved completely.

Assume now the presence of poles on $[b_{2i}, b_{2i+1}]$. Then the contour γ_i contains those poles inside it. Nevertheless, since the image of γ_i under the map

$$z \to R_n^*(z) + (R_n^{*2}(z) - R_n^2)^{1/2}$$

does not pass around the origin the relation $\int_{\gamma_i} f(z)\, dz = 0$ holds. Hence

$$2\int_{\gamma_{i,\varepsilon}} f(x)\, dx + \sum_{k:\, 1/a_{k,n} \in [b_{2i}, b_{2i+1}]} \int_{C_{\varepsilon,k}} f(z)\, dz = 0, \tag{23}$$

where $\gamma_{i,\varepsilon} = [b_{2i}, b_{2i+1}] \setminus \bigcup_{k:\, 1/a_{k,n} \in [b_{2i}, b_{2i+1}]} [1/a_{k,n} - \varepsilon, 1/a_{k,n} + \varepsilon]$ and $C_{\varepsilon,k}$ is a "circumference" with the upper half on the upper sheet and the lower half on the lower sheet of the Riemann surface $\mathscr{R}$.

The substitution $z = 1/a_{k,n} + \varepsilon e^{i\phi}$, $\pi \geqslant \phi \geqslant 0$ for the upper arc and $0 \geqslant \phi \geqslant -\pi$ for the lower arc gives

$$\lim_{\varepsilon \to 0} \sum_{k:\, 1/a_{k,n} \in [b_{2i}, b_{2i+1}]} \int_{C_{\varepsilon,k}} f(z)\, dz = 0.$$

Finally, conditions (18) may be replaced by

$$\text{v.p.} \int_{b_{2i}}^{b_{2i+1}} \left(\kappa_n u_{q-1}(x) + \sum_{j=1}^{l_n} \frac{m_{j,n} a_{j,n}}{1 - a_{j,n} x} h^{1/2}(1/a_{j,n}) \right) \Big/ h^{1/2}(x)\, dx = 0,$$

$i = 1, ...q-1$.

The converse assertion is proved as above.

Let us prove now the equivalence of (18)–(19) to (8)–(12). For that reason take $u_{q-1}(x) = x^{q-1} + c_1 x^{q-2} + \cdots + c_{q-1}$. Then (18) and (19) mean linear dependence of columns in the matrix

$$\begin{pmatrix} e_{ij} & d_i \\ f_{kj} & g_k \end{pmatrix}_{i=1,\, k=1,\, j=1}^{q,\, q-1,\, q-1}.$$

Taking into account linear independence of columns in the matrix $(f_{k,j})_{k=1,\, j=1}^{q-1,\, q-1}$ one proves Lemma 1.

Remark 1. F. Peherstorfer and S. Hölzl [28] proved an analogue of Lemma 1 by slightly different method under the assumption that $1/a_{j,n} \notin [b_1, b_{2q}]$, $j = 1, ..., l_n$.

Remark 2. Other characterizations of regular rows (under different names) may be found in [18, 26, 29].

LEMMA 2. *For any system of q, $q > 1$, intervals*

$$E = [b_1, b_2] \cup \cdots \cup [b_{2q-1}, b_{2q}], \qquad -\infty < b_1 < b_2 < \cdots < b_{2q} < +\infty,$$

there exists a system of $q-1$ circles $K_1, ..., K_{q-1}$, with $K_i \subset \{z\colon \Im z > 0\}$; $K_i \cap K_j = \varnothing$, $i \neq j$; $K_1 = \{z\colon |z-i| = R_1\}$ such that the function

$$x = \Phi(z) = (b_1 - b_2) \prod_{i=0}^{\infty} \frac{(z - T_i(0))^2}{(z - T_i(\xi))(z - T_i(\bar{\xi}))} + b_2, \tag{24}$$

where

$$\prod_i{}' \frac{(T_i(\xi) - \xi)^2 \, T_i^2(\bar{\xi})}{T_i^2(0)(T_i(0) - \xi)^2} = b_1 - b_2, \tag{25}$$

gives the conformal mapping of the region $G(K_1, ..., K_{q-1})$ onto $\mathbb{C} \backslash E$.

Proof. By Köbe's [17] (see also [37, Theorem IX.35]) the region $\mathbb{C} \backslash E$ may be mapped conformally onto a circular domain $T = T^{(0)}$. Without loss of generality it can be assumed that one circle (K_0) coincides with the real axis. Intervals $[-\infty, b_1] \cup [b_{2q}, +\infty]$, $[b_2, b_3]$, ..., $[b_{2q-2}, b_{2q-1}]$ of $\overline{\mathbb{C}} \backslash E$ are mapped then onto q analytic curves $\Gamma_1^{(1)}, ..., \Gamma_q^{(1)}$. The curves $\Gamma_1^{(1)}, ..., \Gamma_q^{(1)}$ connect circles K_i and K_{i+1}, $i = 1, ..., q-2$; the circle K_1 with the real axis K_0, and the circle K_{q-1} with infinity. By the construction there exists an indirectly conformal map $\zeta_1 = \bar{\chi}(\zeta)$ of the domain T onto itself with fixed curves $\Gamma_1^{(1)}, ..., \Gamma_q^{(1)}$. Let us invert $T^{(0)}$ with respect to K_i $(i = 0, 1, ..., q-1)$ and let the inverse be denoted by $T_i^{(1)}$, $i = 0, 1, ..., q-1$. By the Riemann–Schwartz Symmetry Principle the mapping $\bar{\chi}$ may be extended to the domain $T^{(1)} = T^{(0)} \cup T_0^{(1)} \cup \cdots \cup T_{q-1}^{(1)}$. Let the curves $\Gamma_1^{(2)}, ..., \Gamma_{q^2}^{(2)}$ be the inverses of $\Gamma_1^{(1)}, ..., \Gamma_q^{(1)}$ with respect to K_i $(i = 0, 1, ..., q-1)$. Then $\bar{\chi}$ fixes curves $\Gamma_1^{(1)}, ..., \Gamma_q^{(1)}, \Gamma_1^{(2)}, ..., \Gamma_{q^2}^{(2)}$. We invert now $T^{(1)}$ with respect to its boundary circumferences and so on, hence $\bar{\chi}$ is extended on a domain $\Omega = \overline{\mathbb{C}} \backslash F$, where F is the singular set of the corresponding Schottky group $\Gamma(K_1, ..., K_{q-1})$. So the function $f(z) = \chi(z)$ is regular and bounded in $U \backslash F$, where U is a neighborhood of F. Since $\bar{\chi}(z)$ maps each equivalent of K_i under the group $\Gamma(K_1, ..., K_{q-1})$ onto itself, all conditions of Köbe's lemma for $f(z)$ ([37, p. 422]) are satisfied. Hence $\chi(z)$ is a regular and schlicht map of $\overline{\mathbb{C}}$ onto itself and therefore $\chi(z)$ is a Möbius mapping with fixed real axis. Thus $\Gamma_1^{(1)}, ..., \Gamma_q^{(1)}, \Gamma_1^{(2)}, ..., \Gamma_{q^2}^{(2)}, ...$ are parts of a circumference which intersects all circles K_i $(i = 0, 1, ..., q-1)$ under right angles. For the sake of being definite let γ be the imaginary axis and let $z = i$ be the center of K_1.

Let $x=\phi(z)$ be the mapping function. Then by the construction $\phi(0)=b_2$, $\phi(\infty)=b_1$. Let us find this mapping now. Suppose that $b_2=0$. The mapping function $y=\widetilde{\phi}(z)$ has in the fundamental domain G of the group $\Gamma=\Gamma(K_1, ..., K_{q-1})$ one double zero at the origin and two poles at points $z=\xi$ and $z=\bar{\xi}=-\xi$ in the fundamental domain G of the group $\Gamma=\Gamma(K_1, ..., K_{q-1})$. The function $y=\widetilde{\phi}(z)$ has real values on ∂K_0, $\partial K_1, ..., \partial K_{q-1}$, it may be extended by the Riemann–Schwartz Symmetry Principle to the domain $G=\overline{\mathbb{C}}\backslash E$. The extended function will be automorphic with respect to the group Γ since each substitution from Γ is equal to two invertions with respect to ∂K_0 and to ∂K_i or its equivalents. Hence by W. Burnside's theorem [10]

$$\widetilde{\phi}(z)=\frac{\Omega^2(z;0)}{(\Omega(z;\xi))(\Omega(z;\bar{\xi}))}\exp\sum_{k=1}^{q-1} m_k\Phi_k(z), \tag{26}$$

where m_k, $k=1, ..., q-1$ are some integers. It follows from (3), (26) that

$$\widetilde{\phi}(z)=\left(z^2\prod_i{}' \frac{T_i^2(z)(T_i(0)-z)^2}{(T_i(z)-z)^2\,T_i^2(0)}\exp\sum_{k=1}^{q-1} m_k\Phi_k(z)\right)\Bigg/ \left((z-\xi)(z-\bar{\xi})\prod_i\frac{(T_i(z)-\xi)(T_i(z)-\bar{\xi})(T_i(\xi)-z)(T_i(\bar{\xi})-z)}{(T_i(z)-z)^2\,(T_i(\xi)-\xi)(T_i(\bar{\xi})-\bar{\xi})}\right). \tag{27}$$

From the definition of $\Phi_k(z)$ it is evident that $\Delta_{\partial K_i}\arg\Phi_k(z)=2\pi\,\delta_{ik}$, but by the construction of $\widetilde{\phi}(z)$ we have that $\Delta_{\partial K_i}\arg\widetilde{\Phi}(z)=0$ for $i=0, 1, ..., q-1$, hence by (27) it follows that $m_k=0$, $k=1, ..., q-1$. Thus

$$\widetilde{\phi}(z)=\frac{z^2}{(z-\xi)(\xi-\bar{\xi})}\prod_i{}'\frac{T_i^2(z)(T_i(0)-z)^2\,(T_i(\xi)-\xi)(T_i(\bar{\xi})-\bar{\xi})}{T_i^2(0)(T_i(z)-\xi)(T_i(z)-\bar{\xi})(T_i(\xi)-z)(T_i(\bar{\xi})-z)}.$$

Evident relations

$$\frac{(T_i(\xi)-\xi)\,T_i(z)}{(T_i(z)-\xi)\,T_i(\xi)}=\frac{(\xi-T_i^{-1}(\xi))(z-T_i^{-1}(0))}{(z-T_i^{-1}(\xi))(\xi-T_i^{-1}(0))},$$

$$\frac{(T_i(\bar{\xi})-\bar{\xi})\,T_i(z)}{(T_i(z)-\bar{\xi})\,T_i(\bar{\xi})}=\frac{(\bar{\xi}-T_i^{-1}(\bar{\xi}))(z-T_i^{-1}(0))}{(z-T_i^{-1}(\bar{\xi}))(\bar{\xi}-T_i^{-1}(0))},$$

$$\frac{T_i(z)(T_i(0)-z)}{T_i(0)(T_i(z)-z)}=\frac{(T_i^{-1}(0)-z)\,T_i^{-1}(z)}{T_i^{-1}(0)(T_i^{-1}(z)-z)},$$

and (27) prove that

$$\tilde{\phi}(z) = \frac{z^2}{(z-\xi)(z-\bar{\xi})} \prod_i{}' \frac{(T_i^{-1}(0)-z)(T_i(0)-z)(\xi - T_i^{-1}(\xi))}{T_i^{-1}(0)(T_i(\xi)-z)(T_i(\bar{\xi})-z)\, T_i(z)}$$

$$\times \frac{(z-T_i^{-1}(0))^2\,(\bar{\xi}-T_i^{-1}(\bar{\xi}))\, T_i^{-1}(z)(T_i(z)-z)}{\begin{pmatrix} T_i(0)(z-T_i^{-1}(\xi))(\xi-T_i^{-1}(0))(z-T_i^{-1}(\bar{\xi})) \\ (\bar{\xi}-T_i^{-1}(0))(T_i^{-1}(z)-z) \end{pmatrix}}$$

$$= \prod_{i=0}^{\infty} \frac{(z-T_i(0))^2}{(z-T_i(\xi))(z-T_i(\bar{\xi}))}$$

$$\times \prod_i{}' \frac{(T_i^{-1}(0)-z)}{(T_i(0)-z)} \frac{(\xi - T_i^{-1}(\xi))(\bar{\xi}-T_i^{-1}(\bar{\xi}))}{T_i^{-1}(0)\, T_i(z)\, T_i(0)}$$

$$\times \frac{T_i^{-1}(z)(T_i(z)-z)\, T_i(\xi)\, T_i(\bar{\xi})}{(\xi - T_i^{-1}(0))(\bar{\xi}-T_i^{-1}(0))(T_i^{-1}(z)-z)}.$$

Then the equality

$$\frac{(T_i^{-1}(0)-z)(T_i^{-1}(z)-z)\, T_i^{-1}(z)}{(T_i(0)-z)\, T_i(z)(T_i^{-1}(z)-z)} = \frac{T_i^{-1}(0)}{T_i(0)}$$

shows that $\tilde{\phi}(z)$ may be written in the form

$$\tilde{\phi}(z) = \prod_{i=0}^{\infty} \frac{(z-T_i(0))^2}{(z-T_i(\xi))(z-T_i(\bar{\xi}))}$$

$$\times \prod_i{}' \frac{(\xi - T_i^{-1}(\xi))(\bar{\xi}-T_i^{-1}(\bar{\xi}))\, T_i(\xi)\, T_i(\bar{\xi})}{T_i^2(0)(\xi - T_i^{-1}(0))(\bar{\xi}-T_i^{-1}(0))},$$

or

$$\tilde{\phi}(z) = C_2 \prod_{i=0}^{\infty} \frac{(z-T_i(0))^2}{(z-T_i(\xi))(z-T_i(\bar{\xi}))}.$$

Therefore

$$x = \phi(z) = C_2 \prod_{i=0}^{\infty} \frac{(z-T_i(0))^2}{(z-T_i(\xi))(z-T_i(\bar{\xi}))} + b_2.$$

Since $\phi(\infty) = b_1$ we have $C_2 = b_1 - b_2$, and Lemma 2 is proved.

Remark 3. N. I. Achieser [1] was the first who used automorphic functions for approximation theory problems. Moreover he used the conformal mapping of $T^{(0)}$ onto $\bar{\mathbb{C}} \backslash E$ to find Chebyshev polynomials for two intervals in the case of existence of one additional interval such that the Chebyshev polynomial deviated least from zero simultaneously on these three intervals and on given two intervals.

LEMMA 3. *Let $\mathfrak{A}_n$ be regular with respect to E, $q > 1$. Then*

$$R_n^*(x) = R_n g\left(\frac{[z, \xi_0]^{\kappa_n}}{[z; \bar{\xi}_0]^{\kappa_n}} \prod_{i=1}^{l_n} \frac{[z, \xi_i]^{m_{i,n}}}{[z; \bar{\xi}_i]^{m_{i,n}}} \exp\left\{-\sum_{i=1}^{q-1} n_{i+1} \Phi_i(z)\right\}\right) \quad (28)$$

is the solution of problem (1) *with $K = E$. Here $g(y) = (y + 1/y)/2$ is the Joukowski map, $x = \phi(z)$ is the mapping function from Lemma* 2, *$\phi(\xi_i) = 1/a_{i,n}$, $i = 1, ..., l_n$, $\phi(\xi_0) = \infty$, the numbers n_i, $i = 1, ..., q$ are defined by Lemma* 1, *and R_n is found from the relation*

$$\lim_{x \to \infty} \frac{R_n^*(x)}{x^{\kappa_n}} = 1 \Big/ \prod_{i=1}^{l_n} (-a_{i,n})^{m_{i,n}}. \quad (29)$$

Proof. It follows from the regularity and the proof of Lemma 1 that the function $f(x) = R_n^*(x) + ((R_n^*)^2(x) - R_n^2)^{1/2}$ is meromorphic on $\mathbb{C} \setminus E$ with poles at the point ∞ of order κ_n and at points $1/a_{i,n}$ of order $m_{i,n}$, $i = 1, ..., l_n$, if the branch of square roots is chosen by such a way that

$$\lim_{x \to +\infty} (R_n^*(x)\, \omega_n(x))^{1/2}/x^n = 1.$$

By application of the substitution $x = \phi(z)$ into $f(x)$ one obtains the function $y = F(z)$ with poles at ξ_0 of order κ_n and at ξ_i of order $m_{i,n}$, $i = 1, ..., l_n$. Moreover the variation of argument of y under surrounding the circumference ∂K_i is equal to $2n_{i+1}\pi$ and under passing the real axis is $2n_1\pi$. The application of the Riemann–Schwartz Symmetry Principle permits to extend the function $F(z)$ up to an automorphic function with respect to the corresponding Schottky group Γ.

By Burnside's theorem [10]

$$F(z) = C_3 \prod_{i=1}^{l_n} \frac{\Omega^{m_{i,n}}(z; \xi_i)}{\Omega^{m_{i,n}}(z; \bar{\xi}_i)} \frac{\Omega^{\kappa_n - 1}(z; \xi_0)}{\Omega^{\kappa_n - 1}(z; \bar{\xi}_0)} \frac{\Omega(z; \zeta)}{\Omega(z; \xi_0)}, \quad (30)$$

with $\zeta = T_k(\xi_0)$ for an integer k. Since

$$\Omega(z; T_i(\xi)) = \Omega(z; \xi) \exp \Phi_i(z), \qquad i = 1, ..., q-1,$$

one can write down the representation

$$F(z)=C_4\prod_{i=1}^{l_n}\frac{\Omega^{m_{i,n}}(z;\xi_i)}{\Omega^{m_{i,n}}(z;\bar{\xi}_i)}\frac{\Omega^{\kappa_n-1}(z;\xi_0)}{\Omega^{\kappa_n-1}(z;\bar{\xi}_0)}$$

$$\times\exp\left\{\sum_{i=1}^{q-1}k_i\Phi_i(z)\right\},\qquad k_i\in\mathbb{Z}. \tag{31}$$

It follows from (4) that the variation of the argument for the function $\exp\Phi_i(z)$ is equal to zero when z is passing along ∂K_j, $j\neq i$ and equals 2π with z passing ∂K_i counter-clockwise. By (3) the argument of the function

$$\frac{\Omega(z;\xi_i)}{\Omega(z;\bar{\xi})}$$

does not change when z passes the circumference ∂K_j, $j=1,\dots,q-1$.

Next let us transform the equality (31) by using the following auxiliary formulae:

$$\frac{\Omega(z;w)}{\Omega(z;\bar{w})}=\frac{z-w}{z-\bar{w}}\prod_j{}'\frac{(T_j(z)-w)(T_j(\bar{w})-\bar{w})(T_j(w)-z)}{(T_j(z)-\bar{w})(T_j(w)-w)}$$

$$=\frac{z-w}{z-\bar{w}}\prod_j{}'\frac{(T_j(w)-z)(T_j^{-1}(w)-z)(T_j^{-1}(\bar{w})-\bar{w})}{(T_j^{-1}(\bar{w})-z)(T_j^{-1}(w)-w)(T_j(\bar{w})-z)}$$

$$=\prod_{j=0}^{\infty}\frac{z-T_j(w)}{z-T_j(\bar{w})}\prod_j{}'\frac{T_j^{-1}(\bar{w})-\bar{w}}{T_j^{-1}(w)-w}.$$

The result is

$$\prod_{i=1}^{l_n}\frac{\Omega^{m_{i,n}}(z;\xi_i)}{\Omega^{m_{i,n}}(z;\bar{\xi})}=C_5\prod_{i=1}^{l_n}[z,\xi_i]^{m_{i,n}}.$$

From the relation $F(\infty)=1$ the representation

$$F(z)=[z,\xi_0]^{\kappa_n}\prod_{i=1}^{l_n}[z,\xi_i]^{m_{i,n}}\exp\left\{-\sum_{i=1}^{q-1}n_{i+1}\Phi_i(z)\right\}$$

follows. Thus by the definition of $F(z)$ Lemma 3 is proved.

Remark 4. Automorphic functions were used for the first time in approximation theory by N. I. Achieser but his formulae for the case $q=3$, $n_2=1$ in Lemma 3, are different from (24) and (38). The reason is that we use W. Burnside's theorem on the representation of an automorphic function with given zeroes and poles while N. I. Achieser's formula is based on F. Schottky's paper [32].

Remark 5. For $q=2$ it is possible to express all formulae in terms of elliptic functions (see [20] or [27]).

3. MAIN RESULT

Let $K=[b_1, b_2] \cup \cdots \cup [b_{2p-1}, b_{2p}]$, $b_1 \leqslant b_2 < b_3 \leqslant \cdots < b_{2p}$, and if $b_{2i-1}=b_{2i}$, $i=1, ..., p$ then $p>n$. To be definite suppose $b_{2p}=-b_1=1$. From now on let K be fixed. Furthermore, let q be an integer, $1 \leqslant q \leqslant 2p-1$. The collection $\{\mathscr{K}_q, \mathscr{C}_q^{(0)}, \mathscr{N}_q\}$ with $\mathscr{K}_q=\{k_j\}_{j=1}^{q-1} \subset \mathbb{N}$, $1 \leqslant k_j \leqslant p-1$; $\mathscr{C}_q^{(0)} \subset B=\{b_1, b_2, ..., b_{2p}\}$, $|\mathscr{C}_q^{(0)}|=q+1$; $\mathscr{N}_q=\{n_i\}_{i=1}^{q} \subset \mathbb{N}$, is called admissible for $\mathfrak{A}_n$ if the following conditions are satisfied:

1. the sequence $\{k_j\}_{j=1}^{q-1}$ does not decrease and the equation $k_j=k_{j+1}=k_{j+2}$ is not possible for any j, $j=1, 2, ..., q-3$;

2. for any j, $j=1, 2, ..., q-1$ with $k_j \neq k_{j+1}$ exactly one point from b_{2k_j+1}, b_{2k_j} belongs to $\mathscr{C}_q^{(0)}$. Denote it by $\tilde{b}_{2j+1}, \tilde{b}_{2j}$ correspondingly. Furthermore, let us suppose that $b_{2k_j}=\tilde{b}_{2j}$, $b_{2k_j+1}=\tilde{b}_{2j+3}$ for $k_j=k_{j+1}$, and $b_1=\tilde{b}_1 \in \mathscr{C}_q^{(0)}$, $b_{2p}=\tilde{b}_{2q} \in \mathscr{C}_q^{(0)}$;

3. $\sum_{i=1}^{q} n_i=n$, and the equality $n_{j+1}=1$ holds for $k_j=k_{j+1}$.

Moreover, let $k_0=1$, $k_q=p$. Furthermore, consider matrices $\mathfrak{A}^{(j)}$ such that

$$\mathfrak{A}_n \backslash \mathfrak{A}_n^{(j)}=\{a_{i_1^{(j)}, n}, a_{i_2^{(j)}, n}, ..., a_{i_l^{(j)}, n}\} \, (\mathfrak{A}_n^{(0)}=\mathfrak{A}_n),$$

where $1/a_{i_k^{(j)}, n} \in (b_{2m_k^{(j)}}, b_{2m_k^{(j)}+1})$, and for any j the relations $m_i^{(j)} \neq m_k^{(j)}$ with $i \neq k$; $i, k \in \{1, ..., l\}$ hold. Admissible collections for $\{\mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q\}$ for $\mathfrak{A}_n^{(j)}$, $j \neq 0$, are defined as above with conditions:

3′. $\sum_{i=1}^{q} n_i=n-l$, and the equality $n_{j+1}=1$ holds for $k_j=k_{j+1}$;

4. $|\mathscr{C}_q^{(j)}|=q+1+l$, $\{b_{2m_k^{(j)}}, b_{2m_k^{(j)}+1}\} \subset \mathscr{C}_q^{(j)}$, $k=1, 2, ..., l$; $\{m_1^{(j)}, ..., m_l^{(j)}\} \subset \{k_1, ..., k_q\}$.

It is easily to see that for given K and $\mathfrak{A}_n$ there exists a finite number of admissible collections.

Denote by $E(\mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q, \mathfrak{A}_n^{(i)})$ a set $\bigcup_{i=1}^{q} [\tilde{b}_{2i-1}, \tilde{b}_{2i}]$ such that $\tilde{b}_i \notin \mathscr{C}_q^{(j)}$ are defined from the following relations (with $q>1$):

$$\operatorname{rank} \begin{pmatrix} e_{ij} & d_i \\ f_{kj} & g_k \end{pmatrix}_{i=1, k=1, j=1}^{q, q-1, q-1} = q-1, \tag{32}$$

where

$$e_{i,j}=\int_{\tilde{b}_{2i-1}}^{\tilde{b}_{2i}} \frac{x^{q-j-1}}{(-\tilde{h}(x))^{1/2}}\,dx, \qquad i=1, ..., q; \quad j=1, ..., q-1; \tag{33}$$

$$f_{k,j}=\int_{\tilde{b}_{2k}}^{\tilde{b}_{2k+1}} \frac{x^{q-j-1}}{\tilde{h}^{1/2}(x)}\,dx, \qquad k=1, ..., q-1; \quad j=1, ..., q-1; \tag{34}$$

$$\begin{aligned} d_i={}& n_i\pi(-1)^{q-i+1-\tilde{\lambda}_i}+\kappa_n\int_{\tilde{b}_{2i-1}}^{\tilde{b}_{2i}} \frac{x^{q-1}}{(-\tilde{h}(x))^{1/2}}\,dx \\ &+\sum_{j=1}^{l_n} m_{j,n}a_{j,n}\tilde{h}^{1/2}(1/a_{j,n})\int_{\tilde{b}_{2i-1}}^{\tilde{b}_{2i}} \\ &\times\frac{dx}{(1-a_{j,n}x)(-\tilde{h}(x))^{1/2}}, \qquad i=1, ..., q; \end{aligned} \tag{35}$$

$$\begin{aligned} g_k={}&\kappa_n\int_{\tilde{b}_{2k}}^{\tilde{b}_{2k+1}} \frac{x^{q-1}}{\tilde{h}^{1/2}(x)}+\sum_{j=1}^{l_n} m_{j,n}a_{j,n}\tilde{h}^{1/2}(1/a_{j,n}) \\ &\times\int_{\tilde{b}_{2k}}^{\tilde{b}_{2k+1}}\frac{dx}{(1-a_{j,n}x)\,\tilde{h}^{1/2}(x)}, \qquad k=1, ..., q-1; \end{aligned} \tag{36}$$

$$\tilde{h}(x)=\prod_{j=1}^{2q}(x-\tilde{b}_j); \tag{37}$$

$$\tilde{\lambda}_i=\sum_{j=1}^{q}\ \sum_{k:\,(1/a_{k,n})\in(\tilde{b}_{2j},\,\tilde{b}_{2j+1})} m_{k,n}, \qquad \tilde{b}_{2q+1}=+\infty.$$

In denominators in (33)–(36) the square roots are positive and in (35), (37) branch of square root

$$\tilde{h}^{1/2}(1/a_{j,n})$$

is taken the same as in the equality

$$\lim_{z\to\infty}\tilde{h}^{1/2}(z)/z^q=1.$$

It follows from Lemma 1 and the uniqueness of the best approximation polynomial that for any admissible collection $\{\mathcal{K}_q, \mathcal{C}_q^{(j)}, \mathcal{N}_q\}$ and $\mathfrak{A}_n^{(j)}$ there exists at most one set $E(\mathcal{K}_q, \mathcal{C}_q^{(j)}, \mathcal{N}_q, \mathfrak{A}_n^{(i)})$. Now we denote by $\tilde{G}(K_1, ..., K_{q-1})(\mathcal{K}_q, \mathcal{C}_q^{(j)}, \mathcal{N}_q, \mathfrak{A}_n^{(j)})$ a unique domain of the kind from the Introduction such that it is mapped conformally by the function $x=\tilde{\phi}(z)$

from Lemma 2 onto $\mathbb{C}\backslash E(\mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q, \mathfrak{A}_n^{(j)})$. Then by Lemma 3 it is possible to construct the function

$$R_n^*(x) = R_n^*(x, \mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q, \mathfrak{A}_n^{(j)}),$$

which is the Chebyshev–Markov rational function for the set $E(\mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q, \mathfrak{A}_n^{(j)})$. Denote by $\eta_1, ..., \eta_{n+q-1}$ deviation points for $R_n^*(x)$, i.e. $|R_n^*(\eta_i)| = \|R_n^*\|_{C(E(\mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q, \mathfrak{A}_n^{(j)}))}$, $-1 \leqslant \eta_1 < \cdots < \eta_{n_1+1} = \tilde{b}_2 < \eta_{n_1+2} = \tilde{b}_3 < \cdots < \eta_{n+q-l} = 1$.

THEOREM 1. *The solution of problem* (1) *has one of the following forms*:

$$\tilde{R}_n(x) = M_n(x)$$

or

$$\tilde{R}_n(x) = R_n\, g\left(\frac{[z, \xi_0]^{\kappa_n}}{[z; \bar{\xi}_0]^{\kappa_n}} \prod_{i=1}^{l_n} \frac{[z, \xi_i]^{m_{i,n}}}{[z; \bar{\xi}_i]^{m_{i,n}}} \exp\left\{-\sum_{i=1}^{q-1} n_{i+1} \Phi_i(z)\right\}\right)$$

where g is the Joukowski map, $[z; \xi_i]$ and $\exp \Phi_i(z)$ are the Schottky–Burnside functions (4), (5) *constructed from a unique group $\tilde{\Gamma}(\mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q, \mathfrak{A}_n^{(j)})$, such that for any m, $1 \leqslant m \leqslant q$, the set $\bigcup_{i=k_{m-1}}^{k_m} [b_{2i-1}, b_{2i}]$ contains all points η_i, $i = n_1 + n_2 + \cdots + n_{m-1} + m, ... n_1 + n_2 + \cdots + n_m + m$, with possible exclusion of that point from $\tilde{b}_{2l-1}; \tilde{b}_{2l}$, which does not belong to $\mathscr{C}_q^{(j)}$. The constant R_n is defined by the relation*

$$\lim_{x \to +\infty} \frac{\tilde{R}_n(x)(\omega_n(x))}{x^{\kappa_n}} = 1.$$

Proof. Let $\tilde{R}_n(x)$ be the solution of Problem (1). In addition, suppose that $\tilde{R}_n(x)$ is non-degenerate. Then, by Proposition 1, there exist points $\{x_1 < \cdots < x_{n+1}\} \subset K$ such that

$$\tilde{R}_n(x_i) = (-1)^{n-i+1+\sum_{j \in J} s_j} \|\tilde{R}_n\|_{C(K)}.$$

In this case there exists exactly one zero z_i of $\tilde{R}_n(x)$ between any pair of points x_i and x_{i+1}, $i = 1, ..., n$. There is a possibility of existence of deviation points t_j, $j = 1, ..., m$ (i.e. $t_j \neq x_i$, $i = 1, ..., n+1$, and $|\tilde{R}_n(t_i)| = \|\tilde{R}_n\|_{C(K)}$). Suppose that the interval $[b_{2i-1}, b_{2i}]$ contains the points $x_{j_i}, ..., x_{j_{i+1}-1}$. Then there are two cases:

1. the interval (b_{2i}, b_{2i+1}) does not contain points x with

$$|\tilde{R}_n(x)| > \|\tilde{R}_n\|_{C(K)}; \tag{38}$$

2. there are points $x \in (b_{2i}, b_{2i+1})$ where (38) holds.

In the first case one can consider the interval $[b_{2i-1}, b_{2i+2}]$ instead of the intervals $[b_{2i-1}, b_{2i}]$ and $[b_{2i+1}, b_{2i+2}]$ and defines the set $\tilde{K}$ correspondingly. Here $i \in \mathscr{K}_q$, $b_{2i} \notin \mathscr{C}_q^{(0)}$, $b_{2i+1} \notin \mathscr{C}_q^{(0)}$. Then the function $\tilde{R}_n^*(x)$ will be the solution of Problem (1) for $\tilde{K}$. In the second case there are two variants:

(a) there are points x with property (38) between $x_{j_{i+1}-1}$ and $z_{j_{i+1}-1}$ as well as between $z_{j_{i+1}-1}$ and $x_{j_{i+1}}$;

(b) only one from intervals $(x_{k_{i+1}-1}, z_{k_{i+1}-1})$ and $(z_{k_{i+1}-1}; x_{k_{i+1}})$ contains points with (38).

In case (a) there are at least two points $t_{l_i} \in (x_{j_{i+1}-1}, z_{j_{i+1}-1})$ and $t_{l_i+1} \in (z_{j_{i+1}-1}, s_{j_{i+1}})$, and $\tilde{K} = K \cup [t_{l_i}, t_{l_i+1}]$. Here the relations $x_{j_{i+1}-1} = b_{2i}$ and $x_{j_{i+1}} = b_{2i+1}$ hold and the corresponding elements from $(\mathscr{K}_q, \mathscr{C}_q^{0)}, \mathscr{N}_q)$ should be $k_l = k_{l+1} = i$, $\{b_{2i}, b_{2i+1}\} \subset \mathscr{C}_q^{(0)}$, $n_l = 1$, and $n_{l-1} = k_{i+1} - k_i$ (for $b_{2i-2} \in \mathscr{C}_q^0$).

In variant (b) one has to enlarge K in case of need by such a way that for $x \in [b_{2i-1}, \tilde{b}_{2l}]$ and for $x \in [\tilde{b}_{2l+1}, b_{2i+2}]$ relation (38) should not hold, and for $x \in (\tilde{b}_{2l}, \tilde{b}_{2l+1})$ inequality (38) should be satisfied. Here at least one from the equalities $\tilde{b}_{2l} = b_{2i}$ or $\tilde{b}_{2l+1} = b_{2i+1}$ should be satisfied and either $b_{2i} \in \mathscr{C}_q^{(0)}$ or $b_{2i+1} \in \mathscr{C}_q^{(0)}$. Moreover, $k_{l-1} < k_l = i < k_{l+1}$.

Finally we obtain a set $\tilde{K}$ consisting from $q \leqslant 2p-1$ intervals and such that $\tilde{K} = E(\mathscr{K}_q, \mathscr{C}_q, \mathscr{N}_q, \mathfrak{A}_n)$, $\tilde{R}_n(x) = R_n^*(x, \mathscr{K}_q, \mathscr{C}_q^{(0)}, \mathscr{N}_q, \mathfrak{A}_n)$, for some admissible collection $(\mathscr{K}_q, \mathscr{C}_q^{(0)}, \mathscr{N}_q)$, where $\{k_j\}_{j=1}^{q-1}$ are the numbers of gaps from $(b_2, b_3), ..., (b_{2p-2}, b_{2p-1})$ containing gaps $(\tilde{b}_2, \tilde{b}_3), ..., (\tilde{b}_{2q-2}, \tilde{b}_{2q-1})$, n_j is the number of zeros of the function $\tilde{R}_n(x)$ on $[\tilde{b}_{2j-1}, \tilde{b}_{2j}]$, $j = 1, ..., q$, and $\mathscr{C}_q^{(0)}$ contains those points from $b_1, b_2, ..., b_{2p-1}$ which are the ends of intervals forming $\tilde{K}$ (exactly one for each gap).

If $\tilde{R}_n(x)$ is degenerate, then for some $\mathfrak{A}_n^{(j)}$ the function $\tilde{R}_n(x)$ will be nondegenerate solution of problem (1) for $n = n - l$ and

$$\omega_{n-l}^{(j)}(x) = \frac{\omega_n(x)}{\prod_{k=1}^{l} (1 - a_{i_k, n}^{(j)} x)}.$$

Indeed, if $\mathfrak{A}_n^{(j)}$ is regular with respect to K, then the corresponding function $R_n^*(x) = R_n^*(x, \mathscr{K}_q, \mathscr{C}_q^{(j)}, \mathscr{N}_q, \mathfrak{A}_n^{(j)})$, is the Chebyshev–Markov rational function up to constant factor for any $\mathfrak{A}_{n+1}^{(j-1)} = \mathfrak{A}_n^{(j)} \cup \{1/a\}$ with $1/a \in [-1, 1] \backslash K$

$$\left(\tilde{R}_n(x) = \frac{R_n^*(x)(1 - ax)}{a(1 - ax)} \right).$$

Here the points b_i, b_{i+1} such that $b_i < 1/a < b_{i+1}$ are not the alternation points for the row $\mathfrak{A}_n^{(j)}$ but they are for the row $\mathfrak{A}_{n+1}^{(j-1)}$.

Hence the application of Lemmas 1–3 completes the proof.

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REFERENCES

1. N. I. Achyeser, Über einige Funktionen, die in gegebenen Intervallen am wenigsten von Null abweichen, *Bull. Soc. Phys.-Mathem. Kazan. Ser. 3* **3**(2) (1928), 1–69.
2. N. I. Achyeser, Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen, I, *Bull. Acad. Sci. URSS Cl. Sci. Math. Natur. 9* (1932), 1163–1202.
3. N. I. Achyeser, Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen, II, *Bull. Acad. Sci. URSS Cl. Sci. Math. Natur. 3* (1933), 309–344.
4. N. I. Achyeser, Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen, III, *Bull. Acad. Sci. URSS Cl. Sci. Math. Natur. 4* (1933), 449–536.
5. A. I. Aptekarev, Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda lattices, *Math. USSR Sb.* **53** (1986), 233–260.
6. H. Baker, "Abel's Theorem and the Allied Theory," 2nd ed., Cambridge Univ. Press, Cambridge, UK, 1995.
7. S. N. Bernstein, Sur les polynômes orthogonaux relatifs à un segment fini, Part I, *J. Math. Pures Appl.* **9** (1930), 127–177.
8. P. Borwein, T. Erdelyi, and J. Zhang, Chebyshev polynomials and Markov–Bernstein type inequalities for rational spaces, *J. London Math. Soc. (2)* **50** (1994), 501–519.
9. W. Burnside, On a class of automorphic functions, *Proc. London Math. Soc.* **23** (1892), 49–88.
10. W. Burnside, Further note on automorphic functions, *Proc. London Math. Soc.* **23** (1892), 281–295.
11. S. Darlington, Analytical approximations to approximations in the Chebyshev sense, *Bell System Tech. J.* **49** (1970), 1–32.
12. P. J. Davis, "Interpolation and Approximation," Blaisdell, New York, 1966.
13. L. R. Ford, "Automorphic Functions," McGraw–Hill, New York, 1929.
14. L. Fox and I. B. Parker, "Chebyshev Polynomials in Numerical Analysis," Oxford Univ. Press, London, 1968.
15. S. Karlin and W. J. Studden, "Tchebysheff Systems: with Applications in Analysis and Statistics," Interscience, New York, 1966.
16. J. Kluyver, A special case of Dirichlet's problem for two dimensions, *Acta Math.* **21** (1897), 265–286.
17. P. Köbe, Abhandlungen zur Theorie der konformen Abbildung, V, *Math. Z.* **2** (1918), 198–236.
18. M. G. Krein, B. Ya. Levin, and A. A. Nudel'man, On special representation of polynomials that are positive on a system of closed intervals, and some applications, *in* "Functional Analysis, Optimization, and Mathematical Economics: A Collection of Papers Dedicated to the memory of Leonid Vital'evich Kantorovich" (L. J. Leifman, Ed.), pp. 56–114, Oxford Univ. Press, New York, 1990.

19. V. I. Lebedev, Iterative methods of solution of linear operator equations and polynomials deviated least from zero, *in* “Mathematical Analysis and Related Questions of Mathematics: Proceedings of Mathematical Institute” (A. A. Borovkov, Ed.), pp. 89–108, Nauka, Novosibirsk, 1978. [Russian]
20. A. L. Lukashov, “On Chebyshev–Markov problem on two intervals,” Saratov State University, Saratov, 1989; Dep. in VINITI 1.11.1989, No. 6615-V89. [Russian].
21. A. A. Markov, Lectures on functions deviated least from zero, *in* “Selected Papers of A. A. Markov on Continued Fractions and the Theory of Functions Deviated Least from Zero” (N. I. Achieser, Ed.), pp. 244–299, Ogiz, Moscow, 1948. [Russian]
22. F. B. Pakovich, Elliptic polynomials, *Russian Math. Surveys* **50** (1995), 1292–1294.
23. S. Paszkowski, “Numerical Applications of Chebyshev Polynomials and Series,” Nauka, Moscow, 1983. [Russian]
24. F. Peherstorfer, Orthogonal and Chebyshev polynomials on two intervals, *Acata Math. Hungar.* **55** (1990), 245–278.
25. F. Peherstorfer, On Bernstein–Szegö orthogonal polynomials on several intervals, II: Orthogonal polynomials with periodic recurrence coefficients, *J. Approx. Theory* **64** (1991), 123–161.
26. F. Peherstorfer, Orthogonal and extremal polynomials on several intervals, *J. Comput. Appl. Math.* **48** (1993), 187–205.
27. F. Peherstorfer, Elliptic orthogonal and extremal polynomials, *Proc. London Math. Soc. (3)* **70** (1995), 605–624.
28. F. Peherstorfer and S. Hölzl, “Einige Überlegungen zu den verallgemeinerten Tschebisheffpolynomen auf disjunkten Intervallen,” Diplomarbeit, Linz, 1991.
29. I. L. Ptashitzkiĭ, “On Integration in Finite Form of Irrational Differentials,” thesis, Saint-Petersburg, 1881. [Russian]
30. T. J. Rivlin, “Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory,” 2nd ed., Wiley, New York, 1990.
31. V. N. Rusak, “Rational Functions as an Apparatus of Approximation,” Byelorussian Univ. Press, Minsk, 1979. [Russian]
32. F. Schottky, Über einge spezielle Funktion, welche bei einer bestimmten linearen Transformation ungeändert bleibt, *J. Reine Angew. Math.* **101** (1887), 227–272.
33. M. A. Snyder, “Chebyshev Methods in Numerical Approximation,” Prentice–Hall International, Englewoods Cliffs, NJ, 1966.
34. M. L. Sodin and P. M. Yuditskiĭ, Algebraic solution of problems of E. I. Zolotarev and N. I. Achieser on polynomials least deviating from zero, *Teor. Funktsiĭ Funkstional. Anal. i Prilozhen* **56** (1991), 56–64. [Russian]
35. M. L. Sodin and P. M. Yuditskiĭ, Functions deviating least from zero on closed subsets of the real axis, *St. Petersburg Math. J.* **4** (1993), 201–249.
36. P. L. Tchebysheff, Sur les questions de minima qui se rattachent à la représentation approximative des fonctions, *Mém. Acad. St. Petersbourg, Sér. 6* **7** (1859), 199–291.
37. M. Tsuji, “Potential Theory in Modern Function Theory,” Maruzen, Tokyo, 1959.
38. R. K. Vasil’ev, Asymptotics of best uniform approximations on compacts of the real axis for some functions of Lipschitz’s type, *Russian Acad. Sci. Dokl. Math.* **50** (1995), 482–486.